Sliding Mode Control of Chaos in Duffing's Oscillator with Uncertainties

Qin Zhu
Dept. of Mechanical Engineering
Oyama National College of Technology
zhu@oyama-ct.ac.jp

Shan Liang
College of Automation
Chongqin University
lightsun@cqu.edu.cn

***Abstract*—A sliding mode control scheme is proposed to suppress the chaotic response of Duffing's oscillator with unknown forcing function. The excitation is estimated by function approximation with Fourier series. Based on Lyapunov direct method, the adaptive law of the function approximation has been derived. Therefore the sliding mode control can be applied without knowing the bounds of the unknown function. The simulation results are presented to verify the validity of the proposed scheme.**

I. Introduction

Chaotic response is usually undesirable in the engineering practice since its behavior is irregular and unpredictable. Therefore, the control of chaotic systems has received increasing attention in recent years. Several techniques have been applied for the control of a variety of chaotic systems. One approach is based on the sliding mode control (SMC) technique. Konishi *et al* [1] proposed a bang-bang type SMC method to stabilize a class of chaotic systems and showed that the control system is robust to the model uncertainties. Yang *et al* developed a strategy to control Lorenz chaos by applying SMC [2]. Tsai *et al* showed that one could control a chaotic system under external force excitation to arbitrary trajectories, even when the desired trajectories were not located on the embedded orbits of a chaotic system [3]. Yau and Yan presented a sliding mode controller to control Lorenz chaos subject to sector nonlinear input [4]. The proposed control law is robust against both the uncertainty in system parameters and external disturbance.

In the above literatures, it is assumed that the parameters are known or are bounded and the bounds can be expressed with some known functions. However, In a practical situation, the parameters of the system or the bounds may not be given. Especially the disturbance of a system is usually known or difficult to be measured. Furthermore, the parameters of a system may be time varying variables. Thus, under the cases where the unknown parameters exist, a feasible control scheme must be applied in design of the controller.

In this brief note, we consider the adaptive sliding mode control of chaotic continuous time system. Because of the tremendous complexity of chaotic dynamics, the control plant is restricted to the Duffing's oscillator which has been investigated as a benchmark chaotic system.

The Duffing's oscillator or Duffing's equation is of great interest since it has many applications in engineering [5] and the itís control problem has been studied up to now [6]-[8]. In mechanical engineering, such a system can be a model for the motion of a sinusoidally forced beam which is subjected to the electromagnetic force and undergoes large elastic deflections [9][10]. In early research on the control of the Duffing's oscillator, it was found that the PID control could be applied to eliminate the chaotic response [11]. However, the shortcoming of the method is that the proper selection of parameters for the design of the controller is difficulty and the controller has to be adjusted according to the changes of system parameters.

This work investigate the SMC based control configuration for controlling chaotic and regular dynamics in the Duffing's oscillator which forcing term is assumed to be an unknown function. The adaptive controller is configured by combining the function approximation with the SMC. The unknown disturbance is estimated by function approximation with Fourier series, while the sliding mode controller keeps robustness against parameter uncertainty. The purpose of this short paper is to give a new solution for the control problem of Duffing's oscillator by using advanced nonlinear control theory.

II. SMC-BASED CONTROL SCHEME

A. The mathematic model

The fundamental Duffing's oscillator is described by the following equation [12]

$$\ddot{x} + c\dot{x} + dx^3 = A\cos(2\pi f t) \tag{1}$$

Suppose the forcing function $q(t) = A\cos(2\pi f t)$ is unknown. Introducing the state variables $x_1 = x(t)$, $x_2 = \dot{x}(t)$ and rewriting (1) in the state space form yields the controlled Duffing's system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -dx_1^3 - cx_2 + q + u \end{cases} \tag{2}$$

where $u(t)$ is the control input.

It is desired that $\boldsymbol{x}(t)$ be regulated to $\boldsymbol{x}_d = 0$. Thus the trajectory error state is written as

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_d = [x_1 \ \ x_2]^T \tag{3}$$

Before the controller design, the uncertain term with unknown bounds $q(t)$ has to be estimated. Because it is usually a sinusoid function, the Fourier series will be suitable for its approximation.

B. Function approximation

For any set of orthogonal functions $\{z_i(t)\}$ on $[t_1, \ t_2]$, a function $f(t)$, which is a piecewise continuous real-valued function and satisfies Dirichlet conditions, can be represented in terms of $\{z_i(t)\}$ by a series [13]

$$f(t) = \boldsymbol{w}^T\boldsymbol{z}(t) + \varepsilon(t) \tag{4}$$

where

$$\boldsymbol{w} = [a_0 \ a_1 \ b_1 \ a_2 \ b_2 \ \cdots \ a_n \ \ b_n]^T \tag{5}$$

$$\boldsymbol{z}(t) = [1 \ \cos\omega_1 t \ \ \sin\omega_1 t \ \ \cos\omega_2 t \ \ \sin\omega_2 t \ \cdots \cos\omega_n t \ \ \sin\omega_n t]^T \tag{6}$$

and

$$\varepsilon(t) = \sum_{m=n+1}^{\infty} (a_m \cos\omega_m t + b_m \sin\omega_m t) \tag{7}$$

In (5) and (7), the parameters a_0, a_i and b_i are Fourier coefficients. As long as n is large enough, $\varepsilon(t) \to 0$ and $f(t)$ can be expressed as

$$f(t) \approx \boldsymbol{w}^T\boldsymbol{z}(t) \tag{8}$$

Multiplying by $z_i(t)$ in (8), then integrating over the interval $[t_1, \ t_2]$ and using the orthogonality property, we have

$$\int_{t_1}^{t_2} f(t)z_i(t)dt = w_i \int_{t_1}^{t_2} z_i^2(t)dt \tag{9}$$

Hence, the coefficient w_i can be obtained from the quotient

$$w_i = \frac{\int_{t_1}^{t_2} f(t)z_i(t)dt}{\int_{t_1}^{t_2} z_i^2(t)dt} \tag{10}$$

If $\{z_i(t)\}$ is a complete orthogonal set on $[t_1, t_2]$ and the expansion $w_1z_1(t) + w_2z_2(t) + \cdots + w_nz_n(t) + \cdots$ of $f(t)$ converges. Thus

$$\lim_{n\to\infty} \int_{t_1}^{t_2} \left| f(t) - \sum_{i=1}^{n} w_iz_i(t) \right|^2 dt = 0 \tag{11}$$

which implies

$$f(t) \approx \sum_{i=1}^{n} w_iz_i(t) = \boldsymbol{w}^T\boldsymbol{z} \tag{12}$$

Since the time varying function $\boldsymbol{z}(t)$ is defined by (6), the unknown time-varying function $f(t)$ is replaced by a set of unknown constants $\boldsymbol{w} = [w_1 \ w_2 \ \cdots w_n]^T$. Therefore, by suitably choosing the update law of $\boldsymbol{w}$, a good approximation of $f(t)$ can be obtained.

C. Controller design

For the simplicity, we first assume that the unknown disturbance $q(t)$ can be approximated without error and show the sliding control scheme. Then we discuss the effect of the approximation error on the closed loop system. A sliding surface can be defined as [14]

$$s = \lambda x_1 + x_2 \tag{13}$$

then

$$\dot{s} = \lambda x_2 - dx_1^3 - cx_2 + q + u \tag{14}$$

The best approximation of control input $\hat{u}(t)$ of a continuous law that would achieves $\dot{s} = 0$ is

$$\hat{u} = -\left(-dx_1^3 - cx_2 + \hat{q}\right) - \lambda x_2 \tag{15}$$

where $\hat{q}(t)$ is the estimate of forcing term $q(t)$. Choosing control law as

$$u = \hat{u} - \eta s - k\, sgn(s) \tag{16}$$

where the constants $\eta > 0$ and $k > 0$ are parameters to be selected, and $\text{sgn}(s)$ is a function defined by

$$sgn(s) = \begin{cases} +1 & (\text{if } s > 0) \\ 0 & (\text{if } s = 0) \\ -1 & (\text{if } s < 0) \end{cases} \tag{17}$$

Taking (16) into (14), we obtain

$$\dot{s} = \left(q - \hat{q}\right) - \eta s - k\, sgn(s) \tag{18}$$

Applying the function approximation technique described in Section II-B, $q(t)$ and $\hat{q}(t)$ can be presented as

$$q \simeq \boldsymbol{w}^T \boldsymbol{z} \tag{19}$$

and

$$\hat{q} = \hat{\boldsymbol{w}}^T \boldsymbol{z} \tag{20}$$

respectively. The vectors $\boldsymbol{w}$, $\hat{\boldsymbol{w}}$ and $\boldsymbol{z}$ are defined as

$$\begin{aligned} \boldsymbol{w} &= [\, w_0 \;\; w_1 \;\; w_2 \cdots w_{2i}]^T && (21) \\ \hat{\boldsymbol{w}} &= [\hat{w}_0(t)\; \hat{w}_1(t)\; \hat{w}_2(t) \cdots \hat{w}_{2i}(t)]^T && (22) \\ \boldsymbol{z} &= [\, 1 \;\cos\omega_1 t \;\; \sin\omega_1 t \;\; \cos\omega_2 t \\ &\qquad \sin\omega_2 t \;\cdots\; \cos\omega_i t \;\; \sin\omega_i t]^T && (23) \end{aligned}$$

where $(i = 1, 2, \cdots, N)$.

Note $\boldsymbol{w}$ is a constant vector, $\boldsymbol{w} \in R^n$, $\hat{\boldsymbol{w}}(t) \in R^n$ and $\boldsymbol{z} = \boldsymbol{z}(t) \in R^n$ is a vector of basis function, respectively. The integer N can selected large enough such that (20) gives proper function approximation of $q(t)$.

It is clear that if $\hat{\boldsymbol{w}}^T(t)$ is obtained, the control input $u(t)$ can be calculated. Now, we use Lyapunov direct method to find update laws for $\hat{\boldsymbol{w}}^T(t)$ and discuss the stability of the closed loop system. Let

$$\tilde{\boldsymbol{w}} = \boldsymbol{w} - \hat{\boldsymbol{w}} \tag{24}$$

then

$$\dot{\tilde{\boldsymbol{w}}} = -\dot{\hat{\boldsymbol{w}}} \tag{25}$$

Thus, from (18), we have

$$\dot{s} = \tilde{\boldsymbol{w}}^T \boldsymbol{z} - \eta s - k\, \text{sgn}(s) \tag{26}$$

A Lyapunov function candidate can be selected as

$$V = \frac{1}{2}\left(s^2 + \tilde{\boldsymbol{w}}^T \mathbf{Q} \tilde{\boldsymbol{w}}\right) \tag{27}$$

where $\mathbf{Q} \in R^{n \times n}$ is positive definite and symmetric. Taking the time derivative of V along the system trajectory yields

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}\dot{s} = \frac{1}{2}\left(2\tilde{\boldsymbol{w}}^T \mathbf{Q} \dot{\tilde{\boldsymbol{w}}} + 2 s\dot{s}\right) \\ &= s\dot{s} + \tilde{\boldsymbol{w}}^T \mathbf{Q} \dot{\tilde{\boldsymbol{w}}} \end{aligned} \tag{28}$$

Note (26), the first term of right hand of (28) becomes

$$s\dot{s} = -\eta s^2 - k|s| + \tilde{\boldsymbol{w}}^T \boldsymbol{z} s \tag{29}$$

Since $\dot{\tilde{\boldsymbol{w}}} = -\dot{\hat{\boldsymbol{w}}}$, by taking (25) and (29) into (28), it can be further written as

$$\dot{V} = -\eta s^2 - k|s| + \tilde{\boldsymbol{w}}^T \left(\boldsymbol{z} s - \mathbf{Q}\dot{\hat{\boldsymbol{w}}}\right) \tag{30}$$

Because $\eta > 0$ and $k > 0$, if we define update law of $\hat{\boldsymbol{w}}(t)$ as $\left(\boldsymbol{z} s - \mathbf{Q}\dot{\hat{\boldsymbol{w}}}\right) = 0$ or

$$\dot{\hat{w}} = \mathbf{Q}^{-1} z\, s \tag{31}$$

the time derivative of the Lyapunov function becomes

$$\dot{V} = -\eta s^2 - k|s| \le 0 \tag{32}$$

This implies that $\text{V}(t) \le \text{V}(0)$, and therefore, s and $\tilde{\boldsymbol{w}}$ are bounded.

Equation (31) is the rule for updating $\hat{\boldsymbol{w}}$, or alternatively, it is the estimation rule of $\hat{q}(t)$. In numerical simulation, $\hat{\boldsymbol{w}}(t)$ can be calculated by

$$\dot{\hat{\boldsymbol{w}}}_t \simeq \frac{\hat{\boldsymbol{w}}_t - \hat{\boldsymbol{w}}_{t-\Delta t}}{\Delta t} \tag{33}$$

where Δt is the time step in the computation.

D. The approximation error

In Section II-C, two errors are ignored in the procedure of controller design. In this section, we show that the stability of the control system will not be affected if the parameters of the controller are selected suitably.

The first error $\epsilon_1(t)$ is introduced by (19) in which $q(t)$ is expressed by finite terms of Fourier series. The second one $\epsilon_2(t)$ is the approximation error which is generated in updating $\hat{\boldsymbol{w}}$ because it is impossible to have $q(t) = \hat{\boldsymbol{w}}^{\text{T}} z(t)$. Thus the error $\epsilon(t)$ of the control system can be defined as

$$\epsilon = \epsilon_1 + \epsilon_2 \tag{34}$$

Then the forcing function $q(t)$ can be expressed by

$$q = \boldsymbol{w}^T \boldsymbol{z} + \epsilon \tag{35}$$

and the derivative of s in (18) becomes

$$\begin{aligned} \dot{s} &= \left(\boldsymbol{w}^T \boldsymbol{z} - \hat{\boldsymbol{w}}^T \boldsymbol{z}\right) + \epsilon - \eta s - k \,\mathrm{sgn}\,(s) \\ &= \tilde{\boldsymbol{w}}^T \boldsymbol{z} + \epsilon - \eta s - k \,\mathrm{sgn}\,(s) \end{aligned} \tag{36}$$

Since the Lyapunov function candidate is defined by (27), from (36) and (28), the time derivative of V can be derived as

$$\begin{aligned} \dot{V} &= s\left(\tilde{\boldsymbol{w}}^T \boldsymbol{z} + \epsilon - \eta s - k \,\mathrm{sgn}\,(s)\right) + \tilde{\boldsymbol{w}}^T \mathbf{Q}\dot{\tilde{\boldsymbol{w}}} \\ &= -\eta s^2 - k\, s \,\mathrm{sgn}\,(s) + \epsilon s \\ &\quad + \tilde{\boldsymbol{w}}^T \boldsymbol{z} s + \tilde{\boldsymbol{w}}^T \mathbf{Q}\dot{\tilde{\boldsymbol{w}}} \\ &= -\eta s^2 - k|s| + \epsilon s + \tilde{\boldsymbol{w}}^T \left(\boldsymbol{z} s - \mathbf{Q}\dot{\hat{\boldsymbol{w}}}\right) \\ &= -\eta s^2 - k|s| + \epsilon s \end{aligned} \tag{37}$$

Note $k > 0$, we have

$$\dot{V} \leq -\eta s^2 + \epsilon s \leq (-\eta|s| + |\epsilon|)\,|s| \tag{38}$$

Due to the existence of $\epsilon(t)$, we may not determine definiteness of $\dot{V}$ to conclude any stability property of the closed loop system. However, if we choose a suitable set of basis function to make $\epsilon(t)$ as small as possible and select a proper $\eta > 0$, the time derivative of the Lyapunov function becomes $\dot{V} \leq 0$.

E. Numerical simulations

Now, we give the results of numerical simulations to show the effectiveness of the proposed approach. Fourth order Runge Kutta integration method is used to solve the systems of differential equations. In addition, a time step size 0.005 is employed.

The parameters $c = 0.05$, $d = 1$ are selected so that the Duffing's osocillator

$$\ddot{x} + 0.05\,\dot{x} + x^2 = q(t) \tag{39}$$

exhibits a chaotic behavior if no control is applied. The initial states of the system are $x_0(0) = 3.0$, $\dot{x}_0(0) = 4.0$ in the simulations. Suppose $q(t)$

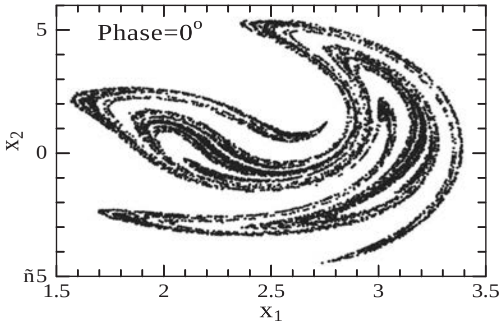

Fig. 1. Poincaré map of the chaotic response.

is the forcing function and its actual expression is

$$q(t) = 7.5\cos(2\pi t) \tag{40}$$

The Poincaré map obtained from the response of the system is shown in Fig. 1 which contains 10000 points and ensures the existence of chaos in the absence of control. The parameters for the controller are selected as

$$\begin{aligned} &\lambda = 1, \quad k = 2, \quad \eta = 9, \quad Q = 0.001I, \\ &\boldsymbol{z} = [1 \;\; \cos t \;\; \sin t \;\; \cos 2t \;\; \sin 2t \\ &\qquad\qquad \cdots \cos 8t \;\; \sin 8t]^T, \\ &\hat{\boldsymbol{w}}_0 = [0.1\; 0.1 \cdots 0.1]^T_{1\times 17} \end{aligned}$$

where $I \in R^{17\times 17}$ is the identity matrix.

Simulation results are shown in Fig. 2. The control started at $t = 40$ s and the chaotic state of the system was suppressed immediately to equilibrium point $(0, 0)$ in presence of the unknown forcing function $q(t)$.

Fig. 2(a) is the time response of state $x(t)$. The sliding behavior of the system is shown in Fig. 2(b). The result of the function approximation for $q(t)$ is shown in Fig. 2(c). The actual function $q(t)$ and the estimated one $\hat{q}(t)$ are shown by the solid and dashed line respectively. The control input is given in Fig. 2(d). The results show that the adaptive sliding mode controller can give good control performance.

III. Conclusion

This work has addressed the adaptive chaos control problem of Duffing's oscillator. A simple adaptive controller has been derived to force the chaotic state to the stable equilibrium point. By

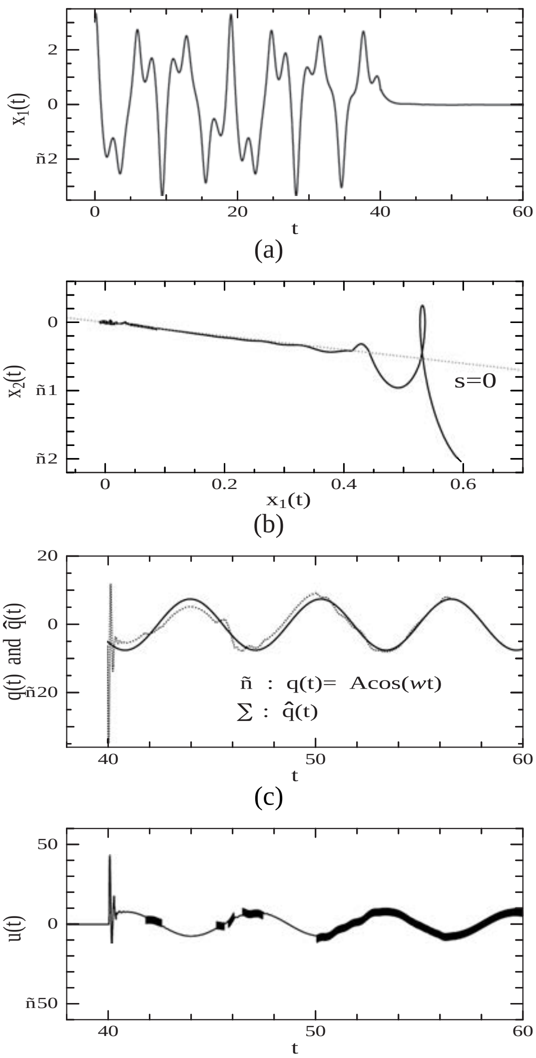

Fig. 2. Control performance. (a) $x_1(t)$ of the Duffing’s oscillator, (b) Sliding behavior, (c) Approximation of $q(t)$, (d) Control input.

function approximation with Fourier series and based on SMC, the controller can be designed without the knowledge of the system disturbance. The stability of the control system is discussed with the Lyapunov stability theorem. The method also applicable for non-chaotic control problem. Numerical simulations are shown to verify the results.

The researches on bifurcation phenomena indicate that, the states of a nonlinear system can jump to the chaotic due to a small variation of the system parameter. In control of a practical system, the exact value of parameters or their bounds are are often not available. Therefore adaptive controller is required. It is hoped that the scheme developed for this particular chaotic system will be applicable to other types of chaotic systems and the control of chaos in the multi-variable system is left for the further study.

REFERENCES

[1] K. Konishi, M. Hirai, H. Kokame, “Sliding mode control for a class of chaotic systems”, *Physics Letters A*, Vol. 245, No. 6 (1998), pp. 511-517.

[2] S.K. Yang, C.L. Chen, H.T. Yau, “Control of chaos in Lorenz system”, *Chaos, Solitons and Fractals*, Vol. 13, No. 4 (2002), pp. 767-780.

[3] H.H. Tsai, C.C. Fuh, C.N. Chang, “A Robust Controller for Chaotic Systems under External Excitation”, *Chaos, Solitons and Fractals*, Vol. 14, No. 4 (2002), pp. 627-632.

[4] H.T. Yau, J.J. Yan, “Design of sliding mode controller for Lorenz chaotic system with nonlinear input”, *Chaos, Solitons and Fractals*, Vol. 19, No. 4 (2004), pp. 891-898.

[5] J.M.T. Thompson and H.B. Stewart, *Nonlinear Dynamics and Chaos*, John Wiley & Sons Inc, 1986.

[6] S. Bowonga, F.M. Moukam Kakmenib, J.L. Dimi, “Chaos control in the uncertain duffing oscillator”, *Journal of Sound and Vibration*, Vol. 292, Issue 3 (2006), pp. 869-880.

[7] R.Wang, J. Deng and Z. Jing, “Chaos control in duffing system”, *Chaos, Solitons and Fractals*, Vol. 27, Issue 1 (2006), pp. 249-257.

[8] Z. Sun, W. Xu, X. Yang and T. Fang, “Inducing or suppressing chaos in a double-well Duffing oscillator by time delay feedback”, *Chaos, Solitons and Fractals*, Vol. 27, Issue 3 (2006), pp. 705-714.

[9] Q. Zhu, D. Gafuka, J. Tani, T. Takagi, K. Otomo, “Chaotic behavior of a cantilever beam subjected to magnetic force” (in Japanese), *Transactions of the Japan Society of Mechanical Engineers*, Vol. 58, No. 551 (1992), pp. 2057-2061.

[10] J. Tani, Q. Zhu, T. Takagi, K. Otomo, “Experiments on chaotic motions of a magnetoelastically buckled beam”, *International Journal of Applied Electromagnetics in Materials*, Vol. 2 (1992), pp. 317-324.

[11] Q. Zhu, J. Tani,T. Takagi, K. Otomo, “Chaotic vibration and its control of a cantilever beam subjected to magnetic force”, *Advanced Computational and Design Techniques in Applied Electromagnetic Systems*, (1995), pp. 333-336.

[12] Y. Ueda, “Random phenomena resulting from nonlinearity in the system described by Duffing’s equation”, Intrnational Journal of Non-linear Mechanics, Vol. 20 (1985), pp. 481-491.

[13] A.C. Huang, Y.K. Kou, “Sliding control of non-linear systems containing time-varying uncertainties with unknown bounds”, *International Journal of Control*, Vol. 74, No. 3 (2001), pp. 252-264.

[14] J.E. Slotine, W. Li, *Applied Nonlinear Control*, Prentice-Hall, Inc, 1991.